ASYMPTOTIC THEORY OF THE TURBULENT BOUNDARY LAYER AS A PROBLEM OF SINGULAR PERTURBATIONS

D. N. Vasil'ev

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We consider an asymptotic theory of the turbulent boundary layer [1,2]. In this paper we make an attempt to further develop the mathematical aspects of this theory. We demonstrate the features of this theory by applying it to a problem which is close to the so-called equilibrium turbulent boundary layer with a pressure gradient and blowing.

NOTATION

x, y	coordinates, parallel and perpendicular to the wall
u	velocity component in the x direction
p, ρ', ν	pressure, density, and kinematic viscosity coefficient
l'	scale of turbulence
τ	tangential stress
u_δ	speed at the outer edge of the boundary layer
δ	thickness of the boundary layer
δ^*	displacement thickness
δ^{**}	momentum loss thickness
C_f	coefficient of friction
R	Reynolds number

The subscript 0 refers to a standard flow; k to a parameter relating to the separation point; * refers to a displacement thickness parameter; ** refers to a momentum loss thickness parameter; and w refers to a parameter relating to the wall.

1. Statement of the Problem, Initial Definitions and Relations. We introduce, following [1], a standard turbulent boundary layer involving the flow of an incompressible fluid over a flat plate, and to describe it we use the subscript 0. For the turbulent tangential stress in the general case, we have

$$\tau = \rho l'^2 (du / dy)^2 \tag{1.1}$$

Introducing the dimensionless coordinate η, the velocity ω, the density ρ, and the scale of turbulence l, we obtain

$$\frac{\tau}{\rho_\delta u_\delta^2} = \rho l^2 \left(\frac{d\omega}{d\eta}\right)^2 \qquad \left(\eta = \frac{y}{\delta}, \omega = \frac{u}{u_\delta}, \rho = \frac{\rho'}{\rho_\delta'}, l = \frac{l'}{\delta}\right) \tag{1.2}$$

We assume that at its outer edge the standard boundary layer has the same parameters as the boundary layer under investigation. From Eq. (1.1) it then follows that

$$\frac{\tau}{\tau_0} = \frac{\rho l^2 (d\omega / d\eta)^2}{\rho_0 l_0^2 (d\omega_0 / d\eta)^2} \tag{1.3}$$

When $\nu \to 0$ the relation (1.3) extends over the whole boundary-layer thickness and, after a formal integration, it assumes the form

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$$1 = \int_0^1 \left(\frac{\rho}{\rho_0} \frac{\tau_0}{\tau}\right)^{1/2} \frac{l}{l_0}\, d\omega \tag{1.4}$$

To an accuracy measured by its satisfaction of Eq. (1.1) the relation (1.4) constitutes a limiting integral formulation relative to the law of friction in the general case. If we take $l \equiv l_0$, then from Eq. (1.4) we obtain

$$1 = \int_0^1 \left(\frac{\rho}{\rho_0} \frac{\tau_0}{\tau}\right)^{1/2} d\omega \tag{1.5}$$

For $R \to \infty$ the integral (1.5) was also obtained in [1] for the most general case of flow over a flat plate. It was shown there that the requirement of self-similarity of l with respect to the perturbation parameters is not necessary in this case for the whole boundary-layer thickness. Only the existence of the condition $l \sim y$ in the region $y_1 < y \ll \delta$ is necessary. The expression (1.4) has an advantage over Eq. (1.5) in that it does not assume conservation of the scale of turbulence in the general case in comparison with a standard flow, and it permits taking this nonconservation readily into account if only a relative quantitative formulation for it is known. The limiting relative friction law may also be written in a different form. From Eq. (1.3), analogous to Eq. (1.4), we obtain

$$\int_0^1 d\omega = \int_0^1 \left(\frac{\rho_0}{\rho} \frac{\tau}{\tau_0}\right)^{1/2} \frac{l_0}{l}\, d\omega_0 \tag{1.6}$$

From Eq. (1.3) we can, in general, obtain several integral formulations of the limiting relative law according as one or another factor appears in the left or the right member of the relation (1.3), this factor being descriptive of the relative perturbation of the standard boundary layer. The limiting relative law can also be formulated in differential form as the boundary-value problem

$$\frac{d\omega}{d\omega_0} = \left(\frac{\rho_0}{\rho} \frac{\tau}{\tau_0}\right)^{1/2} \frac{l_0}{l}, \qquad \begin{array}{lll} \omega = 0 & \text{for} & \omega_0 = 0 \\ \omega = 1 & \text{for} & \omega_0 = 1 \end{array} \tag{1.7}$$

The extra boundary condition gives the desired connection between the relative friction law and the perturbation parameters. A formal integration of Eq. (1.7) yields one or another relative integral law.

We consider the problem involving a turbulent boundary layer in an incompressible liquid with a positive pressure gradient and blowing. To obtain the asymptotic relative friction law from the expressions (1.4), (1.6), or (1.7) in this case, it is necessary to know the relationship between the relative tangential stress and the nonconservation of l and the velocity profiles ω and ω_0 of the boundary layer and the perturbation parameters. In the problem under investigation the perturbations in question are the blowing and the longitudinal pressure gradient. As is done in the majority of the papers in the literature, we assume, as a first approximation, that $l / l_0 \equiv 1$.

The relationship between the tangential stress and the velocity profile ω or ω_0 and the perturbation parameters may be obtained by considering jointly the equations of continuity and motion in the boundary layer. For boundary layers which are close to equilibrium (in the sense that the influence of the derivatives of the perturbation parameters is small in comparison with the influence of these same parameters on the flow in the boundary layer) we can obtain (see, e.g., [3]), from the boundary-layer partial differential equations,

$$\begin{gathered} \frac{\tau}{\rho u_\delta^2} = \frac{C_{f_0}}{2} (\Psi z_1 + b z_2) + (-f) z_3 \\ z_1 = 1 - \frac{\delta}{\delta^{**}} \left(\omega \int_0^\eta \omega\, d\eta - \int_0^\eta \omega^2\, d\eta\right) \\ z_2 = \omega - \frac{\delta}{\delta^{**}} \left(\omega \int_0^\eta \omega\, d\eta - \int_0^\eta \omega^2\, d\eta\right) \\ z_3 = \eta - H \left(\omega \int_0^\eta \omega\, d\eta - \int_0^\eta \omega^2\, d\eta\right) - \omega \int_0^\eta \omega\, d\eta \\ \Psi = \frac{C_f}{C_{f_0}}, \quad b = \frac{2\rho v_w}{\rho u_\delta C_{f_0}}, \quad f = \frac{\delta}{u_\delta} \frac{du_\delta}{dx}, \quad H = \frac{\delta^*}{\delta^{**}} \end{gathered} \tag{1.8}$$

Upon taking these relations into account, we can write the ratio τ/τ_0 in the form

$$\frac{\tau}{\tau_0} = \Psi\varphi_1 + b\varphi_2 + \Lambda\varphi_3 \qquad \left(\varphi_1 = \frac{z_1}{z_{10}},\ \varphi_2 = \frac{z_2}{z_{10}},\ \varphi_3 = \frac{z_3}{z_{10}},\ \Lambda = \frac{-2f}{C_{f_0}}\right) \tag{1.9}$$

For a qualitative analysis of the problem we can find the distribution of τ/τ_0 by using a polynomial approximation. In this case [1]

$$\tau/\tau_0 = \Psi + (b\omega + \Lambda\eta) f(\eta) \tag{1.10}$$

Upon approximating the tangential stress by a second or third degree polynomial, the expression $f(\eta)$ assumes, respectively, the form

$$f(\eta) = (1 + \eta)^{-1}, \quad \text{or} \quad f(\eta) = (1 + 2\eta)^{-1}$$

In solving the problem an analytical formulation for the standard velocity profile is also necessary. Both theoretical solutions, based on similarity and dimensionality considerations, as well as the numerous experimental data classified in [4], show that the total turbulent universal velocity-defect law

$$\frac{1-\omega_0}{\sqrt{{}^1/_2 C_{f_0}}} = D_0(\eta), \qquad D_0(\eta) = -\frac{1}{\varkappa}[\ln\eta + W(\eta)] \tag{1.11}$$

In [1] it was shown that for $R \to \infty$ the thickness of the viscous sublayer $\eta_1 \to 0$ and that

$$\sqrt{{}^1/_2 C_{f_0}} = -\varkappa(\ln\eta_1)^{-1} \to 0 \tag{1.12}$$

Taking this circumstance into account, we find from Eq. (1.11) that

$$\omega = 1 \quad \text{for} \quad \eta > 0, \qquad \omega_0 = 0 \quad \text{for} \quad \eta = 0 \tag{1.13}$$

Thus the representation (1.11) for $R \to \infty$ is correct in the sense that it satisfies the boundary condition at the wall. Various quantitative formulations of the trace function $W(\eta)$ are available depending essentially on how the boundary-layer thickness is defined.

In this paper, for the thickness of the turbulent boundary layer under standard conditions, we take that value of the transverse coordinate for which the correlation coefficient of the turbulent velocity fluctuations assumes the value zero,

$$K = \frac{\langle uv\rangle}{(\langle u^2\rangle\langle v^2\rangle)^{1/2}}$$

This coordinate also corresponds to a zero value of the turbulent tangential stress. We approximated the trace function by the expression

$$W = a_0 + a_1\eta + {}^1/_2 a_2\eta^2 + \int_0^\eta \eta^{-a_3}\exp\left(\frac{-a_4}{\eta}\right)d\eta \tag{1.14}$$

In addition, in determining the coefficients a_i, we assumed, from the condition for smooth coupling with the external flow, that $D_0 \approx 0$ and $dD_0/d\eta \approx 0$. The three remaining coefficients were determined by the method of least squares from the experimental data for D_0 given in [4]. Moreover, to satisfy the condition of equality of the correlation coefficient at the external edge of the boundary layer to zero, we found it necessary to increase its thickness by a factor of 1.2 compared with that used in [4]. This is plainly evident in Fig. 1 where we have used Klebanov's experimental data (see, for example, [3]) for the distribution of the correlation coefficient and the relative tangential stress for one of the experiments, appearing also in the series treated in [4].

The lower scale of abscissas in this figure corresponds to the boundary-layer thickness adopted in [4]. Determination of the turbulent boundary-layer thickness from the correlation coefficient is more precise inasmuch as this function approaches the value zero rather steeply, whereas in the region $y \approx \delta$ the variation of the velocity is very small. The approximation of $D_0(\eta)$ in accord with Eq. (1.14), along with the experimental data, is shown in Fig. 2. In what follows we assume that the universality of the law (1.11), observed in [4] for finite numbers R, continues to hold as $R \to \infty$. In Fig. 1 we also display the theoretical relationship for determining the tangential stress, obtained from the expressions (1.8) and (1.14) for $dp/dx = 0$ and $R \to \infty$.

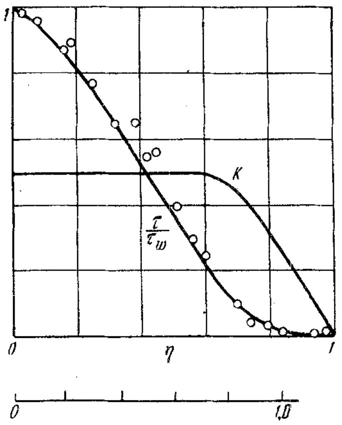

Fig. 1

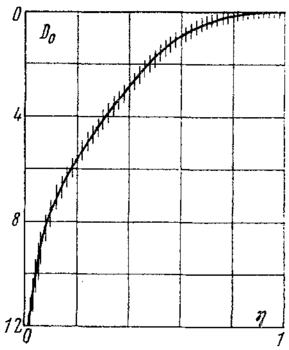

Fig. 2

From a joint consideration of the expressions (1.7), (1.9), and (1.11) it follows that the problem concerning the asymptotic relative friction law and the asymptotic velocity profile leads, in the given case, to the solution of the integrodifferential equation

$$\left(\frac{d\omega}{d\eta}\right)^2 = \left\{\frac{C_{f_0}}{2}\left[\Psi\varphi_1 + b\varphi_2\right] + (-f)\varphi_3\right\}\left(-\frac{dD_0}{d\eta}\right)^2 \tag{1.15}$$

with the boundary conditions (1.7), and with one formulation or another for the functions φ_1, φ_2, φ_3, D_5.

2. Asymptotic Relative Friction Law and Asymptotic Velocity Profile. A general solution of Eq. (1.15), even in the case involving the simplest approximating representation (1.10) of the functions φ_1, φ_2, φ_3 is not known. For $R \to \infty$, the friction coefficient $C_{f_0} \to 0$, and if we neglect the term containing C_{f_0}, we obtain

$$\left(\frac{d\omega}{d\eta}\right)^2 = (-f)\,\varphi_3\left(\frac{dD_0}{d\eta}\right)^2 \tag{2.1}$$

For this equation the boundary condition $\omega = 0$ for $\eta = 0$ cannot be satisfied in the general case. As was shown in [5], in such cases one suspects the presence of the singular nature of the solution, and the classical method of neglecting a term with a small parameter or expanding in a series with respect to this parameter can no longer be applied for the entire domain of the solution. To solve such problems the method of matched asymptotic expansions was proposed in [5]. In accord with this method we introduce an internal and an external region of the solution and we expand this solution in a series in terms of the parameter $\gamma_0 = \sqrt{C_{f_0}/2}$ We consider the external solution (superscript e) in the form

$$\omega = \omega^e = \sum \omega_n{}^e \gamma_0{}^n \tag{2.2}$$

Substituting Eq. (2.2) into Eq. (1.15) and equating terms with identical powers of γ_0, we obtain, to within γ_0,

$$\frac{d\omega_0{}^e}{d\eta} = \sqrt{(-f)\varphi_3}\left(-\frac{dD_0}{d\eta}\right), \qquad \omega_0{}^e = 1 \quad \text{for} \quad \eta = 1 \tag{2.3}$$

Here the superscript e, together with the subscript 0, denotes the external solution of zero order with respect to γ_0. A formal integration with respect to η gives

$$\omega_0 = 1 - \sqrt{(-f)}\int_\eta^1 \sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta \tag{2.4}$$

For the internal solution we introduce the superscript i and a new internal variable, for which we choose the dimensionless velocity under standard conditions

$$\omega_0 = 1 - \gamma_0 D_0 \tag{2.5}$$

Expanding in a series with respect to γ_0, we write the internal solution in the form

$$\omega^i = \sum \omega_n{}^i(\omega_0)\gamma_0{}^n \tag{2.6}$$

Substituting this relation into Eq. (1.15) and equating terms with identical exponents of γ_0, we obtain, to within γ_0,

$$(d\omega_0{}^i / d\omega_0)^2 = \Psi\varphi_1 + b\varphi_2, \qquad \omega_0{}^i = 0 \quad \text{for} \quad \omega_0 = 0 \tag{2.7}$$

Here the superscript i, together with the subscript 0, denotes the internal solution of zero order with respect to γ_0. For small η (the domain of the internal solution), it follows from Eqs. (1.8) and (1.9), or from the initial boundary-layer partial differential equations, that

$$\varphi_1 \to 1, \qquad \varphi_2 \to \omega \qquad \text{for} \qquad \eta \to 0$$

Taking this circumstance into account, we obtain from Eq. (2.7)

$$(d\omega_0^i / d\omega_0)^2 = \Psi + b\omega_0^i, \qquad \omega_0^i = 0 \qquad \text{for} \qquad \omega_0 = 0 \tag{2.8}$$

The solution (2.8) has the form

$$\omega_0^i = \sqrt{\Psi}\,\omega_0 + {}^1/_4 b\omega_0^2 \tag{2.9}$$

The parameter Ψ is as yet free, its connection with f and b being established through the coalescence of the internal and external solutions. To see what this connection is, we apply the principle of matched limits, formulated as follows [5]: the external limit of the internal expansion is equal to the internal limit of the external expansion.

The analytical formulation of this principle in our case has the form

$$\omega_0^i(1) = \omega_0^e(0) \tag{2.10}$$

Substituting Eqs. (2.4) and (2.9) into Eq. (2.10), we obtain

$$\sqrt{\Psi} + {}^1/_4 b = 1 - \sqrt{(-f)}\int_0^1 \sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta \tag{2.11}$$

This is, in fact, the limiting relative friction law for gradient flow with blowing. For the velocity profile we can construct a composite solution of zero order in γ_0 over the whole domain of the solution. We use the additive method [5] to obtain the composite solution

$$\omega_0^+ = \begin{cases} \omega_0^e + \omega_0^i - \omega_0^e(0) \\ \omega_0^e + \omega_0^i - \omega_0^i(1) \end{cases} \tag{2.12}$$

Here the plus superscript, together with the subscript 0, denotes the composite solution of zero order in γ_0. Taking into account Eqs. (2.4), (2.9), and (2.11), we obtain from Eq. (2.12)

$$\omega_0^+ = 1 - \sqrt{(-f)}\int_\eta^1 \sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta + \sqrt{\Psi}\,\omega_0 + \frac{b}{4}\omega_0^2 - \sqrt{\Psi} - \frac{b}{4}$$

or

$$\omega_0^+ = \sqrt{(-f)}\int_0^\eta \sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta + \sqrt{\Psi}\,\omega_0 + \frac{b}{4}\omega_0^2 \tag{2.13}$$

Eliminating Ψ with the aid of Eq. (2.11), we obtain

$$\omega_0^+ = \sqrt{(-f)}\left[\int_0^\eta \sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta - \omega_0\int_0^1 \sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta\right] + \omega_0\left(1 - \frac{b}{4}\right) + \frac{b}{4}\omega_0^2 \tag{2.14}$$

From the expressions (1.13) and (2.13) it is clearly evident that for $\gamma_0 \to 0$ the solution consists of two parts: a singular part

$$\sqrt{\Psi}\,\omega_0 + {}^1/_4 b\omega_0^2$$

concentrated entirely in an infinitesimally small η neighborhood, and a regular part, defined in the interval $0 < \eta \leq 1$. Taking Eq. (1.13) into account, we can write the solution (2.13) for $R \to \infty$ also in the form

$$\omega_0^+ = \left(\sqrt{\Psi} + \frac{b}{4}\right) + \sqrt{(-f)}\int_0^\eta \sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta \quad \text{for} \quad \eta > 0$$
$$\omega_0^+ = 0 \quad \text{for} \quad \eta = 0 \tag{2.15}$$

Thus an application of the principle of matched asymptotic expansions has enabled us to exhibit in a clear and simple manner the singular nature of the problem of the limiting flow ($R \to \infty$, $\gamma_0 \to 0$) in a turbulent boundary layer with a pressure gradient and blowing.

We compare the solution (2.15) with several exact particular solutions of Eq. (1.15).

The case of flow with a pressure gradient in the absence of blowing. In this case the differential equation of the problem takes the form

$$\frac{d\omega}{d\eta} = \sqrt{\gamma_0^2\Psi\varphi_1 + (-f)\varphi_3}\left(-\frac{dD_0}{d\eta}\right) \tag{2.16}$$

with the boundary conditions (1.7).

The solution of this equation may be written in the form

$$\omega = \lim_{\substack{\eta_1\to 0\\ \eta_2\to 0}} \int_{\eta_1}^{\eta_2} \frac{1}{\varkappa\eta}\sqrt{\gamma_0^2(\eta_1)\Psi + (-f)\eta}\, d\eta + \int_{\eta_2}^{\eta} \sqrt{\gamma_0^2(\eta_1)\Psi\varphi_1 + (-f)\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta \tag{2.17}$$

where $\eta_2 > \eta_1$ and account has been taken of the fact that

$$\left(-\frac{dD_0}{d\eta}\right) \to \frac{1}{\varkappa\eta} \qquad \text{for} \qquad \eta \to 0$$

A very essential fact here is that the parameter γ_0 is connected with the lower limit of integration in the first integral. Calculating the first integral, we obtain

$$\omega = \lim_{\substack{\eta_1\to 0\\ \eta_2\to 0}} \frac{\gamma_0(\eta_1)\sqrt{\Psi}}{\varkappa} \ln\frac{(\Phi_2-\theta)(\Phi_1+\theta)}{(\Phi_2+\theta)(\Phi_1-\theta)} + \frac{2}{\varkappa}(\Phi_2 - \Phi_1)$$
$$+ \int_0^{\eta} \sqrt{(-f)\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta \qquad (\eta_2 > \eta_1) \tag{2.18}$$
$$\Phi_\nu = \sqrt{\gamma_0^2(\eta_1)\Psi + (-f)\eta_\nu} \qquad (\nu = 1, 2), \qquad \theta = \sqrt{\gamma_0^2(\eta_1)\Psi}$$

Substituting

$$\gamma_0^2(\eta_1) = \left(-\frac{\varkappa}{\ln\eta_1}\right)^2 \qquad \text{for} \qquad \eta_1 \to 0$$

into the expression (2.18) and evaluating the limit on the right-hand side, we obtain, after transformations have been made,

$$\omega = \sqrt{\Psi} - \sqrt{(-f)}\int_0^{\eta}\sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta \qquad \text{for} \qquad \eta > \eta_1 \to 0 \tag{2.19}$$
$$\omega = 0 \quad \text{for} \quad \eta = \eta_1 \to 0$$

Thus we obtain the same result as was obtained previously by the method of matched asymptotic expansions for $b = 0$.

The case of a flow with blowing in the absence of a pressure gradient. In this case Eq. (1.15) assumes the form

$$\frac{d\omega}{d\eta} = \sqrt{\gamma_0^2(\Psi\varphi_1 + b\varphi_2)}\left(-\frac{dD_0}{d\eta}\right) \tag{2.20}$$

or after introducing the variable ω_0

$$d\omega / d\omega_0 = \sqrt{\Psi\varphi_1 + b\varphi_2} \tag{2.21}$$

In the region of the wall $\eta \to 0$, $\varphi_1 \to 1$, $\varphi_2 \to \omega$ and, consequently,

$$d\omega / d\omega_0 = \sqrt{\Psi + b\omega} \tag{2.22}$$

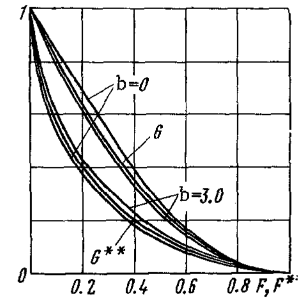

Fig. 3

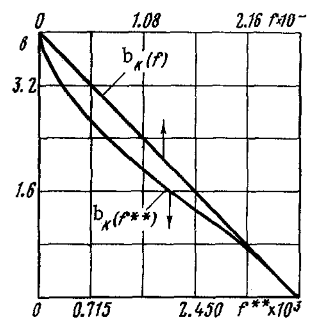

Fig. 4

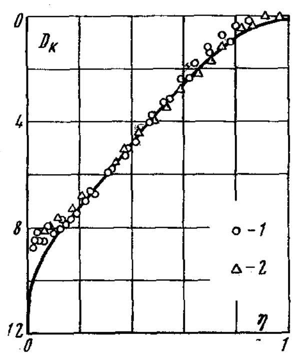

Fig. 5

The solution of this equation with the boundary conditions (1.7) has the form [1]

$$\omega = \sqrt{\Psi}\,\omega_0 + {}^1\!/_4 b\omega_0^2 \tag{2.23}$$

Substituting Eq. (2.23) into Eq. (1.8), we find that if $\gamma_0 \to 0$, then $z_1 \to z_{10}$ for $0 \le \eta \le 1$, $\varphi_2 \to 1$ for $\eta > 0$, $\varphi_2 \to \omega$ for $\eta \to 0$.

Consequently for $\gamma_0 \to 0$ and $f \equiv 0$, the equation (2.21) goes over into Eq. (2.22) and, together with its solution (2.23), does not depend on η. Then just as in the preceding case, the solution (2.23) coincides with the solution (2.13), obtained by the method of matched asymptotic expansions.

3. Numerical Results. We consider at first the case in which the function φ_3 is self-similar in the perturbation parameters, chosen, for example, in accord with Eq. (1.10). In this case it follows from Eq. (2.11) that

$$\sqrt{\Psi} = (1 - {}^1\!/_4 b)(1 - \sqrt{F}), \qquad F = f/f_k \tag{3.1}$$

The parameter f_k is also obtained from Eq. (2.11) subject to the condition $\Psi = 0$

$$\sqrt{(-f_k)} = \left(1 - \frac{b}{4}\right)\left[\int_0^1 \sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right)d\eta\right]^{-1} \tag{3.2}$$

From Eqs. (3.1) and (3.2) it is evident that the limiting relative friction law in the form (3.1) does not depend on a specific quantitative formulation for φ_3. The specific form of this function determines only the parameter f_k for given b, or the parameter b_k for given f. From the relations (1.8) we see that the functions φ_1, φ_2, φ_3 depend implicitly on the perturbation parameters, and an arbitrary method, based on their self-similarity in regard to the perturbations, does not give correct quantitative results. We extract from Eq. (2.15) the singular part of the solution; this leads to a change in the boundary condition for the velocity at the wall. In this case the equations (2.15), (1.8), and (1.9) form the system

$$\frac{d\omega}{d\eta} = \sqrt{-f}\,\sqrt{\varphi_3}\left(-\frac{dD_0}{d\eta}\right), \quad \frac{dJ_1}{d\eta} = \omega, \quad \frac{dJ_2}{d\eta} = \omega^2 \tag{3.3}$$

where

$$\varphi_3 = \frac{\eta - H(J_{10} - J_2) - \omega J_1}{1 - [D_0\eta - J_3(\eta)][J_3(1)]^{-1}}, \quad J_1 = \int_0^\eta \omega\, d\eta, \quad J_2 = \int_0^\eta \omega^2\, d\eta, \quad J_3 = \int_0^\eta D_0\, d\eta$$

The boundary conditions assume the form

$$\begin{aligned} &\omega = \sqrt{\Psi} + {}^1\!/_4 b, \ J_1 = 0, \ J_2 = 0 \quad \text{for} \quad \eta = 0 \\ &\omega = 1, \ J_1 = 1 - \delta^*/\delta, \ J_2 = 1 - \delta^*/\delta - \delta^{**}/\delta \quad \text{for} \quad \eta = 1 \end{aligned} \tag{3.4}$$

The boundary conditions at $\eta = 1$ enable us to find Ψ, δ^*/δ, δ^{**}/δ, and H as functions of f and b.

The system (3.3) with the boundary conditions (3.4) was solved on a digital computer.

In Fig. 3 we have plotted the limiting relative friction law in the form

$$G(F, b) = \frac{\Psi}{(1 - {}^1\!/_4 b)^2} \tag{3.5}$$

The relationship between the parameters b and f at the separation point of the boundary layer, $\Psi = 0$, is shown in Fig. 4. In the turbulent boundary-layer calculations a more suitable gradient parameter is one formulated in terms of the momentum loss thickness

$$f^{**}=\frac{\delta^{**}}{u_\delta}\frac{du_\delta}{dx}$$

Therefore, in Fig. 3 we also plot the limiting relative friction law in the form

$$G^{**}(F,b)=\frac{\Psi^{**}}{(1-{}^1/_4 b)^2},\quad F^{**}=\frac{f^{**}}{f_k^{**}} \tag{3.6}$$

The relationship between the parameters b and f^{**} at the boundary-layer separation point ($\Psi^{**}=0$) is shown in Fig. 4. We see that the functional relations (3.5) and (3.6) exhibit weak stratification with respect to the parameter b; they may be satisfactorily approximated by the expressions

$$G=(1-F)^2,\quad G^{**}=1-\sqrt{F^{**}(2-F^{**})} \tag{3.7}$$

In the absence of a pressure gradient ($f=0$) the results obtained here are identical to the theoretical results obtained in [1], wherein a fairly detailed comparison of theory with experiment was made. In [6], based on a careful analysis of a large amount of contemporary experimental data relating to separation of the turbulent boundary layer, it was established that $2.2 < H_k < 2.8$, the values $H_k > 2.4$ specifying, most likely, the secondary flows or distortions due to probes inserted near the wall. In this paper we obtain the theoretical value $H_k = 2.3$. In Fig. 5 we compare the limiting calculated velocity profile $D_k = (1-\omega)/\sqrt{(-f^*)}$ for $\Psi = 0$ with known experimental profiles due to Stratford [7] with $\Psi \approx 0$. (We use the symbol O for data at Sec. 1 and Δ for data at Sec. 2). Experimental values of the gradient parameters obtained therein were $0.0065 < f_k^* < 0.01$, $H_k \approx 2.29$.

The theoretical value $f_k^* = f^{**}H_k = 0.00821$, and also the theoretical velocity profile, agree satisfactorily with the experimental values.

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